Quantum analogues of geometric inequalities for Information Theory

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based on a joint work with Robert Koenig and Stefan Huber

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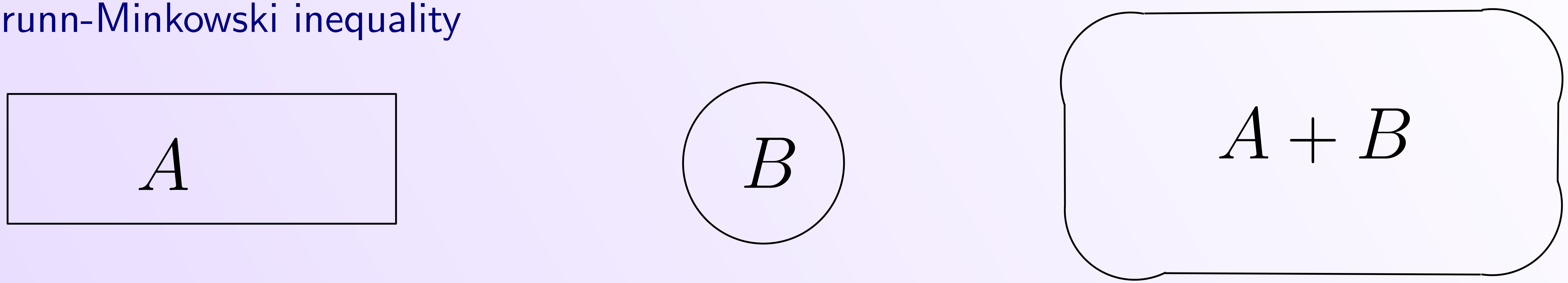
Outline of the talk

- Introduction
  - Geometric inequalities in information theory
  - Classical vs quantum inequalities
- Quantum inequalities
  - Entropy power inequality
  - Concavity of the entropy power of diffusion semigroup
  - Quantum isoperimetric inequality
- Application to the quantum Ornstein-Uhlenbeck semigroup
- Optimizing entropy rates of the quantum attenuator semigroup

Geometric inequalities

$\mathbb{R}^n = \mathbb{R}^2$

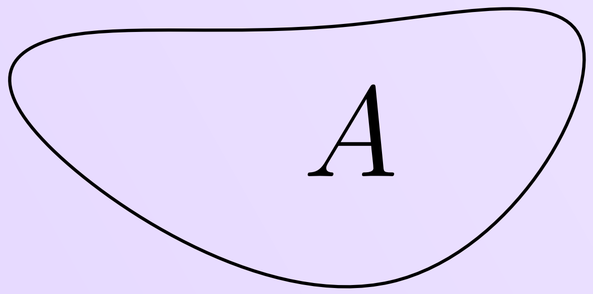
Brunn-Minkowski inequality



$A + B = \{a + b | a \in A, b \in B\}$

$$\mathrm{vol}(A)^{1/n} \quad + \quad \mathrm{vol}(B)^{1/n} \quad \leq \quad \mathrm{vol}(A+B)^{1/n}$$

Isoperimetric inequality



$$\mathrm{area}(A) \leq \frac{1}{4\pi}\mathrm{length}(\partial A)^2$$

Geometry vs Information theory

Set $A \subset \Omega$

volume $\mathrm{vol}(A)$

Random variable X on Ω
with prob. mass function $p_x = \Pr[X = x]$

entropy power $2^{H(X)}$

Shannon entropy $H(X) = -\sum p_x \log p_x$

Geometry vs Information theory

Set $A \subset \Omega$

volume $\mathrm{vol}(A)$

Consider a set $A^n = A \times \cdots \times A \in \Omega^n$

$\mathrm{vol}(A^n) = (\mathrm{vol}(A))^n$

Random variable X on Ω
with prob. mass function $p_x = \Pr[X = x]$

entropy power $2^{H(X)}$

Shannon entropy $H(X) = -\sum p_x \log p_x$

Consider n i.i.d. with $P_{X^n} = P_X \cdot \cdots \cdot P_X$

has the following property:

$\forall \epsilon > 0 \qquad \exists M_{n,\epsilon} \subset \Omega^n$ s.t.

$|M_{n,\epsilon}| \sim \left(2^{H(X)}\right)^n$

and $\Pr[X^n \in M_{n,\epsilon}] \geq 1 - \epsilon$

Geometry vs Information theory

Set $A \in \mathbb{R}^n$

Random variable X on $\mathbb{R}^n$
with prob. density function f_X

volume $\mathrm{vol}(A)$

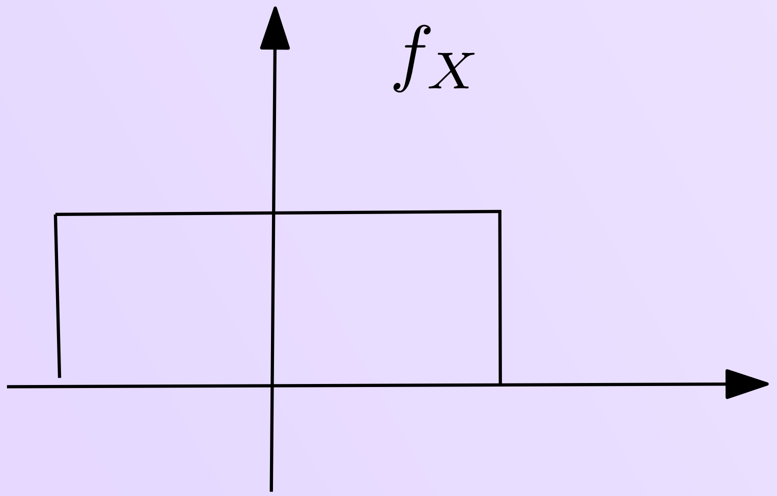
entropy power $e^{2H(X)/n}$

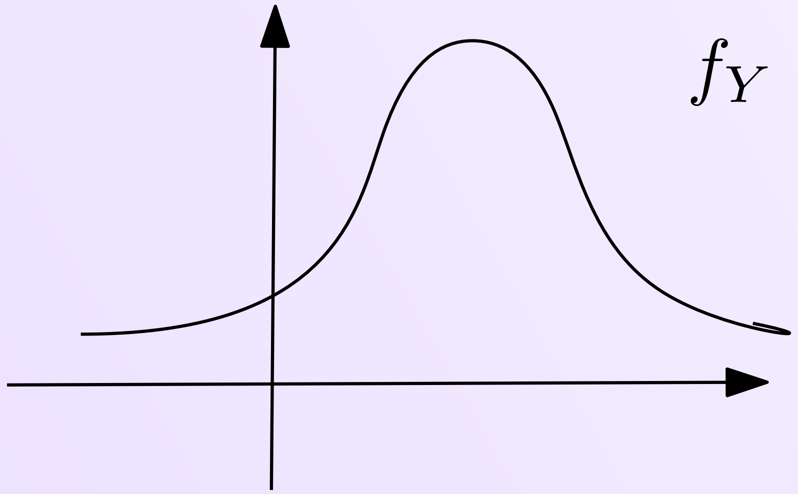
entropy $H(X) = -\int_{\mathbb{R}^n} f_X(x) \log f_X(x) dx$

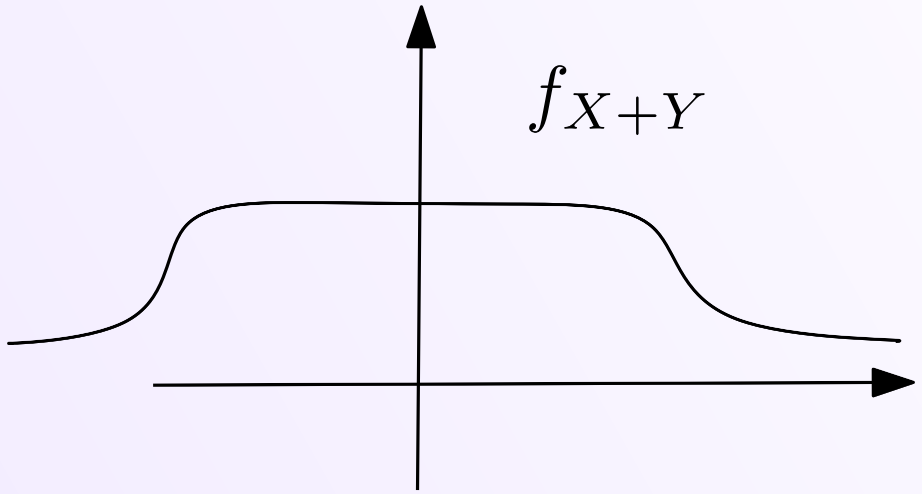
addition $A + B$

convolution: $X + Y$ has a density function

$f_{X+Y}(x) = \int f_X(x-z) f_Y(z) dz$







Geometric inequalities

Brunn-Minkowski inequality

$$\mathrm{vol}(A)^{1/n} + \mathrm{vol}(B)^{1/n} \leq \mathrm{vol}(A+B)^{1/n}$$

Shannon's entropy power inequality ['48]

$$e^{2H(X)/n} + e^{2H(Y)/n} \leq e^{2H(X+Y)/n}$$

Geometric inequalities

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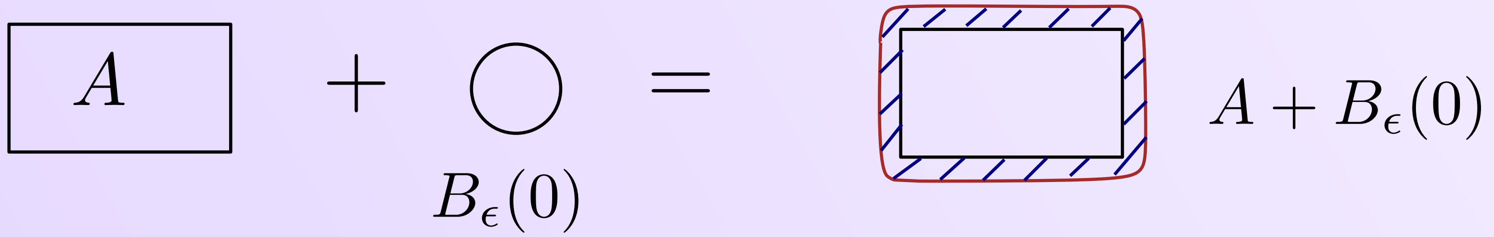
Isoperimetric inequality

$$\mathrm{area}(A) \leq \frac{1}{4\pi}\mathrm{length}(\partial A)^2$$

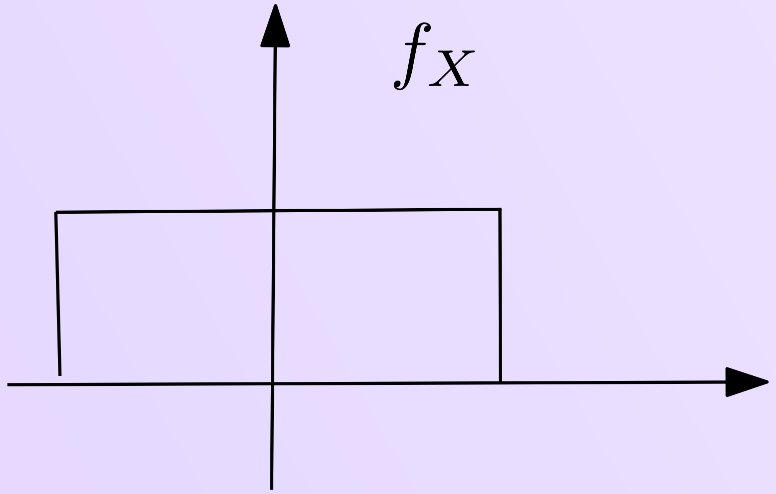
$$\mathrm{length}(\partial A) = \lim_{\epsilon\to 0} \frac{\mathrm{area}(A+B_\epsilon(0)) - \mathrm{area}(A)}{\epsilon}$$

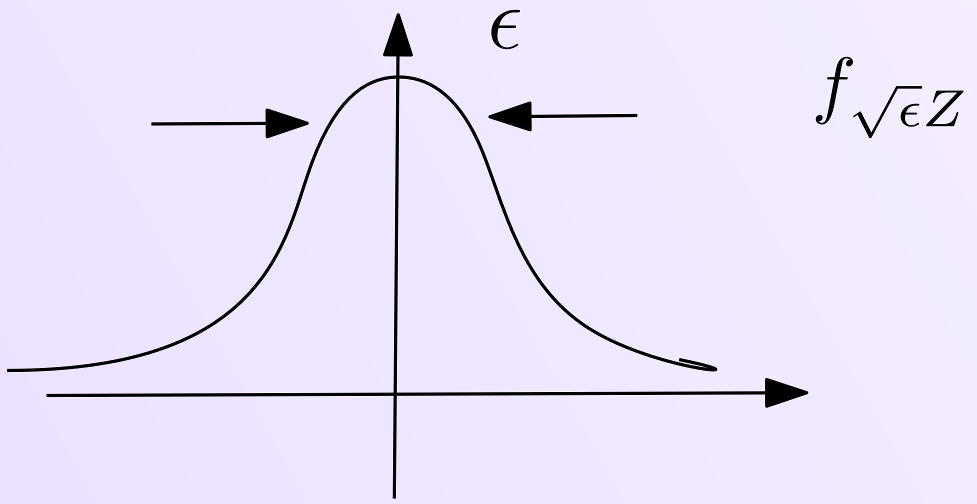


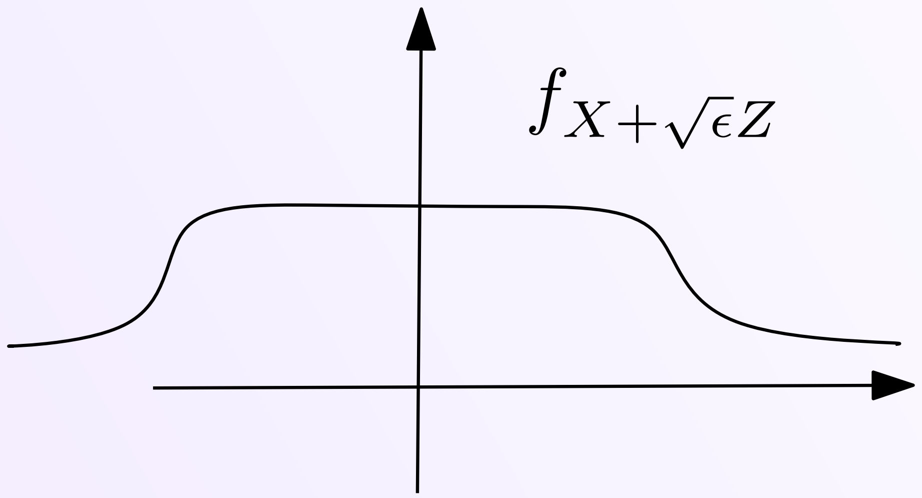
Shannon's entropy power inequality ['48]

$$e^{2H(X)/n} + e^{2H(Y)/n} \leq e^{2H(X+Y)/n}$$

"length" $\lim_{\epsilon\to 0} \frac{e^{2H(X+\sqrt{\epsilon}Z)/n} - e^{2H(X)/d}}{\epsilon}$







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Geometric inequalities

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$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} e^{2H(X+\sqrt{\epsilon}Z)/n} = \frac{2}{n} e^{2H(X)/n} \; \frac{d}{d\epsilon}\Big|_{\epsilon=0} H(X+\sqrt{\epsilon}Z)$$

Fisher information

$$J(X) = 2\,\frac{d}{d\epsilon}\Big|_{\epsilon=0} H(X+\sqrt{\epsilon}Z)$$

Geometric inequalities

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Shannon's entropy power inequality ['48]

$$e^{2H(X)/n} + e^{2H(Y)/n} \leq e^{2H(X+Y)/n}$$

Isoperimetric inequality for entropies

$$\tfrac{1}{n} J(X) e^{2H(X)/n} \geq 2\pi\, e$$

"length" $\lim_{\epsilon\to 0} \frac{e^{2H(X+\sqrt{\epsilon}Z)/n} - e^{2H(X)/d}}{\epsilon}$

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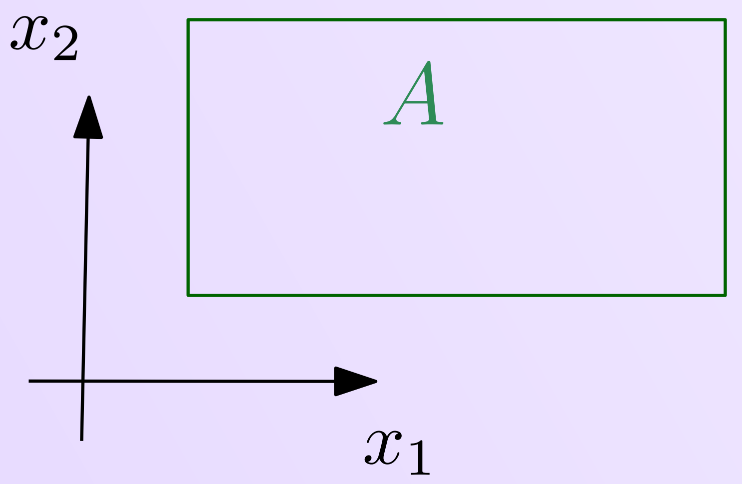
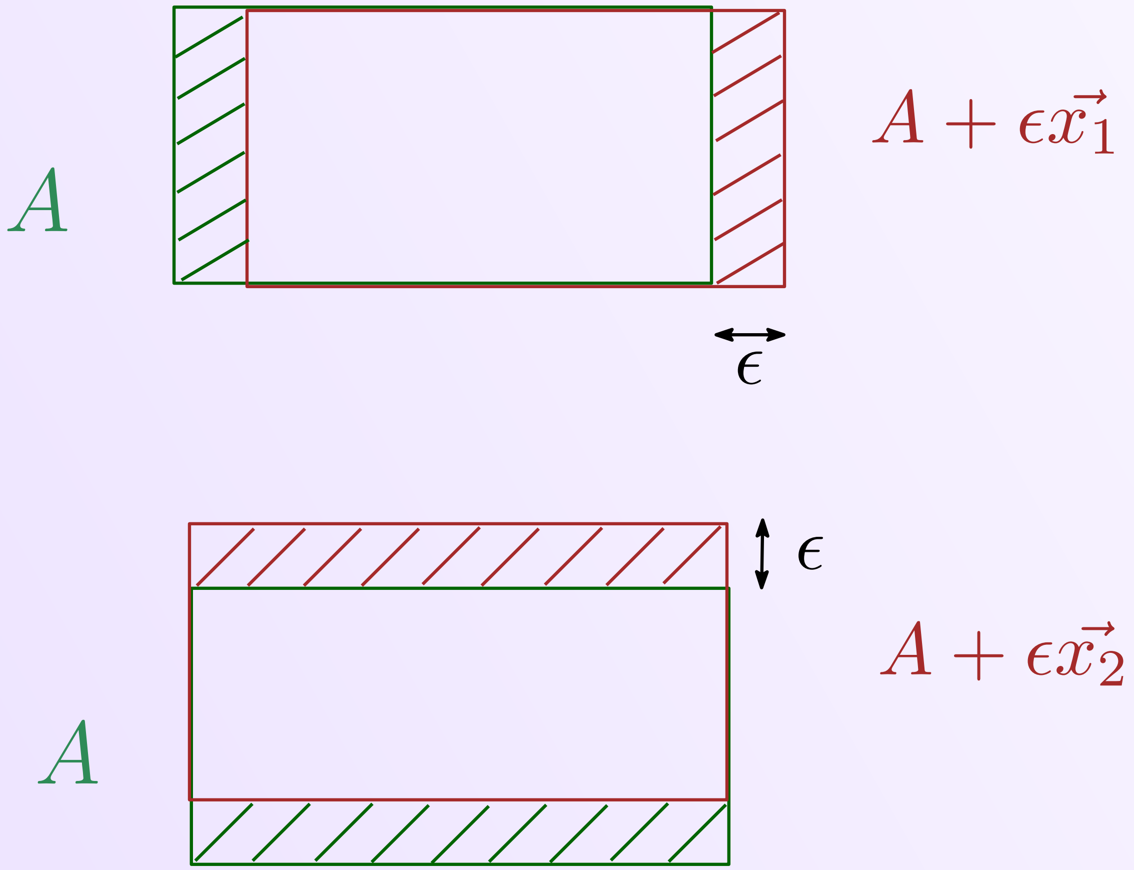
Fisher information

$$J(X) = 2\, \frac{d}{d\epsilon}\Big|_{\epsilon=0} H(X+\sqrt{\epsilon}Z)$$

Geometric analogue of Fisher information

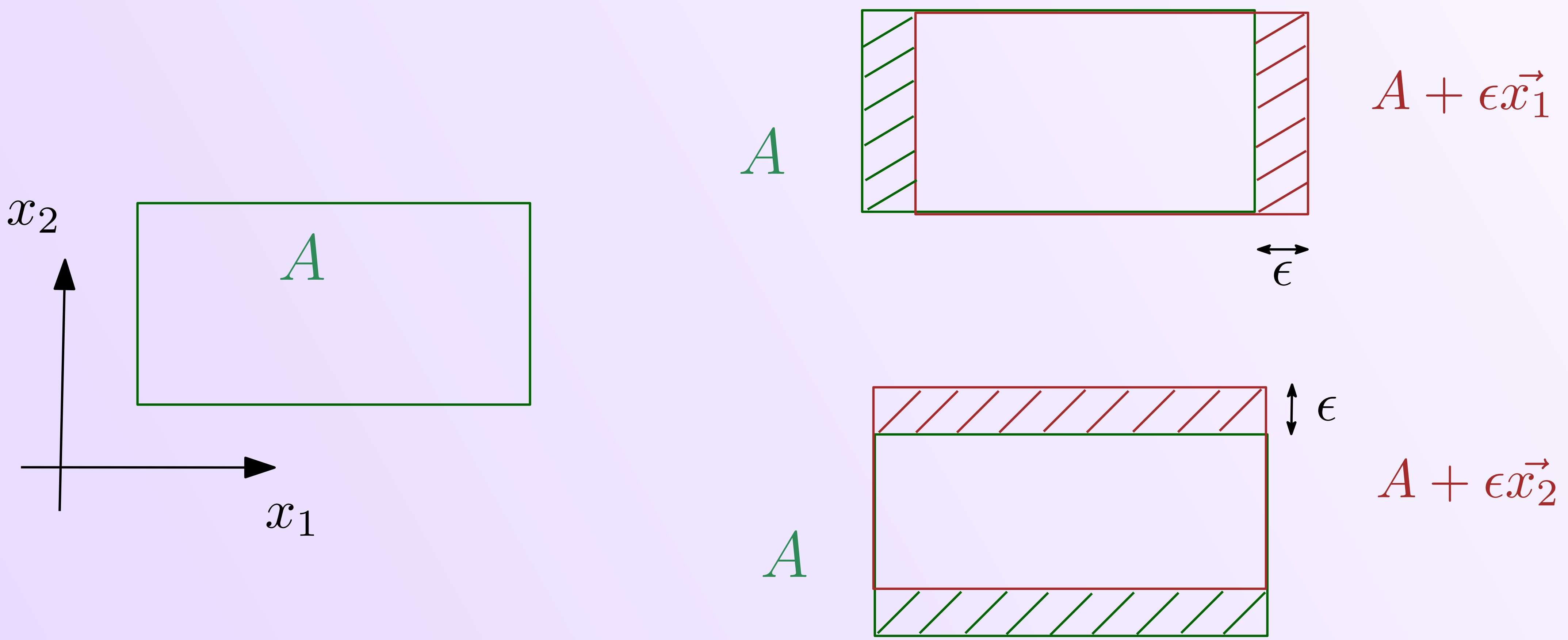




$D(A_1, A_2) =$ "difference" between A_1 and A_2

"Fisher Information" $= \frac{d^2}{d\epsilon^2}\Big|_{\epsilon=0} \sum_{j=1}^{2} D(A, A + \epsilon \vec{x_j})$

Geometric analogue of Fisher information



distribution translated in $\vec{x_j}$ direction by ϵ

$D(A_1, A_2) =$ "difference" between A_1 and A_2

$$J(X) = \sum_{j=1}^{n} \frac{\partial^2}{\partial \epsilon^2}\Big|_{\epsilon=0} D(f || f^{\epsilon \vec{x_j}})$$

de Bruijn's identity

"Fisher Information" $= \frac{d^2}{d\epsilon^2}\Big|_{\epsilon=0} \sum_{j=1}^{2} D(A, A + \epsilon \vec{x_j})$

$$= 2 \frac{d}{d\epsilon}\Big|_{\epsilon=0} H(X + \sqrt{\epsilon} Z)$$

Geometry	Classical	Quantum
Set $A \in \mathbb{R}^n$	Random variable X on $\mathbb{R}^n$ with prob. density function f_X	ρ - n-mode state. $[Q_j, P_k] = -[P_k, Q_j] = i\delta_{j,k} I$ $[Q_j, Q_k] = [P_j, P_k] = 0 \qquad 1 \leq j, k, \leq n$
volume $\mathrm{vol}(A)$	entropy power $e^{2H(X)/n}$	$N(\rho) = \exp\{S(\rho)/n\}$
	$H(X) = -\int_{\mathbb{R}^n} f_X(x) \log f_X(x) dx$	$S(\rho) = -\mathrm{Tr}(\rho \log \rho)$
addition $A + B$	convolution: $X + Y$	
	$f_{X+Y}(x) = \int f_X(x-z) f_Y(z) dz$	

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addition $A + B$	convolution: $X + Y$	$\rho_X + \rho_Y$??
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addition $A + B$	convolution: $X + Y$	$\rho_{X \boxplus_\lambda Y} = \mathrm{Tr}_Y \left(U_\lambda (\rho_X \otimes \rho_Y) U_\lambda^\dagger \right)$
	$f_{X+Y}(x) = \int f_X(x - z) f_Y(z) dz$	beamsplitter with transmissivity $\lambda \in [0, 1]$ $U_{\lambda,1}^\dagger Q_{1(2)} U_{\lambda,1} = \sqrt{\lambda} Q_{1(2)} + \sqrt{1-\lambda} Q_{2(1)}$ $U_{\lambda,1}^\dagger P_{1(2)} U_{\lambda,1} = \sqrt{\lambda} P_{1(2)} + \sqrt{1-\lambda} P_{2(1)}$ $U_{\lambda,n} = U_{\lambda,1}^{\otimes n}$

Geometry

Set $A \in \mathbb{R}^n$

volume $\mathrm{vol}(A)$

addition $A + B$

Brunn-Minkowski inequality

$$\mathrm{vol}(A)^{1/n} + \mathrm{vol}(B)^{1/n} \leq \mathrm{vol}(A+B)^{1/n}$$

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$$S(\rho) = -\mathrm{Tr}(\rho \log \rho)$$

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beamsplitter with transmissivity $\lambda \in [0,1]$

Quantum entropy power inequality

$$\lambda e^{S(X)/n} + (1-\lambda) e^{S(Y)/n} \leq e^{S(X \boxplus_\lambda Y)/n}$$

[Koenig, Smith 14] $\lambda = 1/2$

[de Palma et al 15]

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$$S(\rho) = -\mathrm{Tr}(\rho \log \rho)$$

classical-quantum

$$f \star_t \rho = \int f(\xi) W(\sqrt{t}\xi) \rho W^\dagger(\sqrt{t}\xi) d\xi$$

Geometry	Classical	Quantum
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addition $A + B$	convolution: $X + Y$	classical-quantum
	$f_{X+Y}(x) = \int f_X(x - z) f_Y(z) dz$	$f \star_t \rho = \int f(\xi) W(\sqrt{t}\xi) \rho W^\dagger(\sqrt{t}\xi) d\xi$
Brunn-Minkowski inequality	Shannon entropy power inequality	Classical-quantum entropy power inequality
$\mathrm{vol}(A)^{1/n} + \mathrm{vol}(B)^{1/n} \leq \mathrm{vol}(A+B)^{1/n}$	$e^{2H(X)/d} + e^{2H(Y)/d} \leq e^{2H(X+Y)/d}$	$te^{H(f)/n} + e^{S(\rho)/n} \leq e^{S(f \star_t \rho)/n} \qquad t \geq 0$
Isoperimetric inequality	Isoperimetric inequality for entropies	Isoperimetric inequality for entropies
$\mathrm{area}(A) \leq \frac{1}{4\pi} \mathrm{length}(\partial A)^2$	$\frac{1}{n} J(X) e^{2H(X)/n} \geq 2\pi e$	$\frac{1}{n} J(\rho) e^{S(\rho)/n} \geq 4\pi e$

Classical vs Quantum information theory

Classical

X - $\mathbb{R}^n$-valued r.v. with a prob. density function f_X.

Entropy

Shannon

$$H(X) = H(f) = -\int f(x)\log f(x)dx$$

Entropy Power $N(f) = \exp\{2H(f)/n\}$

Relative entropy $D(f||g) = \int f(x)\log\frac{f(x)}{g(x)}dx$

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von Neumann

$$S(\rho) = -\mathrm{Tr}(\rho\log\rho)$$

$$N(\rho) = \exp\{S(\rho)/n\}$$

$$D(\rho||\sigma) = \mathrm{Tr}(\rho\log\rho - \rho\log\sigma)$$

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Fisher information matrix For $\vec{\theta} \in \mathbb{R}^n$, define

$$f^{(\theta)}(x) = f(x-\theta)$$

$$J(f^{(\theta)})\big|_{\theta=\theta_0} = \left(\frac{\partial^2}{\partial\theta_i\partial\theta_j}\Big|_{\theta=\theta_0} D\left(f^{(\theta_0)}||f^{(\theta)}\right)\right)_{i,j=1}^{2n}$$

Fisher information

$$J(f) = \mathrm{Tr}\left(J(f^{(\theta)})\big|_{\theta=\theta_0}\right)$$

Quantum

ρ - n-mode state.

$[Q_j, P_k] = -[P_k, Q_j] = i\delta_{j,k}I$

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von Neumann

$$S(\rho) = -\mathrm{Tr}(\rho\log\rho)$$

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Quantum Fisher information

For $\vec{\theta} \in \mathbb{R}^{2n}$ the Weyl displacement operators are defined as

$$W(\vec{\theta}) = \exp\{i\sqrt{2\pi}\,(\theta_1 P_1 - \theta_2 Q_1 + \cdots + \theta_{2n-1} P_n - \theta_{2n} Q_n)\}$$

Translated state is

$$\rho^{(\theta)} = W(\theta)\rho W^{\dagger}(\theta)$$

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Weyl operators translate position and momentum operators

$$W(\vec{\theta}) Q_j W^\dagger(\vec{\theta}) = Q_j + \theta I \qquad W(\vec{\theta}) P_j W^\dagger(\vec{\theta}) = P_j + \theta I$$

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Fisher information matrix

$$J(\rho^{(\theta)})\big|_{\theta=\theta_0} = \left(\frac{\partial^2}{\partial\theta_i \partial\theta_j}\Big|_{\theta=\theta_0} D\left(\rho^{(\theta_0)} || \rho^{(\theta)}\right) \right)_{i,j=1}^{2n}$$

Fisher information

$$J(\rho) = \mathrm{Tr}\left(J(\rho^{(\theta)})\big|_{\theta=\theta_0} \right)$$

Inequalities

Classical

$$(f_X, f_Y) \to f_{X+Y}(z) = \int f_X(z-x) f_Y(x) dx$$

Quantum

$$(f, \rho) \to f \star_t \rho = \int f(\xi) W(\sqrt{t}\xi) \rho W^\dagger(\sqrt{t}\xi) d\xi$$

$t = 1$ [Werner ‘84]

Inequalities

Classical

$$(f_X, f_Y) \to f_{X+Y}(z) = \int f_X(z-x) f_Y(x) dx$$

- the Fisher information ineq.

  for $\lambda \in [0,1]$

  $$J\left(\sqrt{\lambda}X + \sqrt{1-\lambda}Y\right) \leq \lambda J(X) + (1-\lambda) J(Y)$$

- Stam inequality

  $$J(X+Y)^{-1} \geq J(X)^{-1} + J(Y)^{-1}$$

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$t = 1$ [Werner '84]

- the Fisher information ineq.

  for $\omega = \sqrt{t}\omega_c + \omega_q$

  $$\omega^2 J(f \star_t \rho) \leq \omega_c^2 J(f) + \omega_q^2 J(\rho)$$

- Stam inequality

  $$J(f \star_t \rho)^{-1} \geq t J(f)^{-1} + J(\rho)^{-1}$$

Inequalities

Classical

$(f_X, f_Y) \to f_{X+Y}(z) = \int f_X(z-x) f_Y(x) dx$

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Z - Gaussian r.v. : $f_Z(\vec{\xi}) = (2\pi)^{-n/2} e^{-|\xi|^2/2}$

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Quantum Diffusion Semigroup

For Z - Gaussian r.v. : $f_Z(\vec{\xi}) = (2\pi)^{-n} e^{-|\xi|^2/2}$

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$$= e^{t\mathcal{L}}(\rho)$$

for a Liouvillean

$$\mathcal{L}(\rho) = -\pi \sum_{j=1}^{2n} [Q_j, [Q_j, \rho]] + [P_j, [P_j, \rho]]$$

The state satisfies

$$\rho(t) = e^{t\mathcal{L}}(\rho) : \qquad \frac{d}{dt}\rho(t) = \mathcal{L}(\rho(t))$$

Inequalities

Classical

Quantum

$$e^{t\mathcal{L}}(\rho) = f_Z \star_t \rho$$

- de Bruijn identity

$$J\left(X+\sqrt{t}Z\right) = 2\frac{\partial}{\partial t}H\left(X+\sqrt{t}Z\right)$$

$$J(\rho) = 2\frac{d}{dt}S\left(e^{t\mathcal{L}}(\rho)\right)\Big|_{t=0}$$

[Koenig, Smith '14]

- Concavity of the entropy power

$$\frac{d^2}{d\epsilon^2}\bigg|_{\epsilon=0}\{N(X+\sqrt{\epsilon}Z)\} \leq 0$$

$$\frac{d^2}{dt^2}\bigg|_{t=0} N(e^{t\mathcal{L}}(\rho)) \leq 0$$

- Fisher information isoperimetric inequality

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\left\{\left[\frac{1}{n}\{J(X+\sqrt{\epsilon}Z)\}\right]^{-1}\right\} \geq 1$$

$$\frac{d}{dt}\bigg|_{t=0}\left\{\left[\frac{1}{2n}J(e^{t\mathcal{L}}(\rho))\right]^{-1}\right\} \geq 1$$

- Entropy power inequality

$$N(X+Y) \geq N(X) + N(Y)$$

$$N(f \star_t \rho) \geq tN(f) + N(\rho)$$

- Isoperimetric inequality

$$\frac{1}{n}J(X)N(X) \geq 2\pi e$$

$$\frac{1}{n}J(\rho)N(\rho) \geq 4\pi e$$

Quantum Inequalities

Rescaling

If f_X is a prob. density function of a r.v. X, then

$$f_X \star_t \rho = f_{\sqrt{t}X} \star_1 \rho$$

here the r.v. $\sqrt{t}X$ is given by $f_{\sqrt{t}X}(\zeta) = f(\zeta/\sqrt{t})/(\sqrt{t})^{2n}$

Addition

$$f_X \star_1 (f_Y \star_1 \rho) = f_{X+Y} \star_1 \rho$$

where $(f_X, f_Y) \to f_{X+Y}(z) = \int f_X(z-x) f_Y(x) dx$

Translation

Let $\omega_c, \omega_q > 0$, and $t \geq 0$. Then for all $\theta \in \mathbb{R}^{2n}$

$$(f \star_t \rho)^{(\omega\theta)} = f^{(\omega_c \theta)} \star_t \rho^{(\omega_q \theta)} \qquad \omega = \sqrt{t}\omega_c + \omega_q$$

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Data Processing Inequality

$$D\left(f \star_t \rho \| g \star_t \sigma\right) \leq D\left(f \| g\right) + D\left(\rho \| \sigma\right)$$

Stam Inequality

Theorem

Let $\omega_c, \omega_q \in \mathbf{R}$, and $t \geq 0$. Then

$$\omega^2 J(f \star_t \rho) \leq \omega_c^2 J(f) + \omega_q^2 J(\rho) \qquad \text{for } \omega = \sqrt{t}\omega_c + \omega_q$$

In particular,

$$J(f \star_t \rho)^{-1} - J(\rho)^{-1} - tJ(f)^{-1} \geq 0$$

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Proof

By Data processing inequality

$$\mathrm{Tr}\left(J(f^{(\omega_c \vec{\theta})} \star_t \rho^{(\omega_q \vec{\theta})}); \vec{\theta}) \Big|_{\vec{\theta}=\vec{\theta}_0} \right) \leq \mathrm{Tr}\left(J(f^{(\omega_c \vec{\theta})}; \vec{\theta}) \Big|_{\vec{\theta}=\vec{\theta}_0} \right) + \mathrm{Tr}\left(J(\rho^{(\omega_q \vec{\theta})}; \vec{\theta}) \Big|_{\vec{\theta}=\vec{\theta}_0} \right)$$

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$$\mathrm{Tr}\left(J(\{f \star_t \rho\}^{(\omega\vec{\theta})}; \vec{\theta}) \Big|_{\vec{\theta}=\vec{0}} \right)$$

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Let $\omega_c, \omega_q \in \mathbf{R}$, and $t \geq 0$. Then

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$$\omega^2 J(f \star_t \rho)$$

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$$\mathrm{Tr}\left(J(\{f \star_t \rho\}^{(\omega\vec{\theta})}; \vec{\theta}) \Big|_{\vec{\theta}=\vec{0}} \right)$$

$$J(f^{(\omega_c\vec{\theta})}; \vec{\theta}) = \omega_c^2 J(f^{(\vec{\theta})}; \vec{\theta})$$

$$\omega^2 J(f \star_t \rho)$$

$$\omega_c^2 J(f)$$

Stam Inequality

Theorem

Let $\omega_c, \omega_q \in \mathbf{R}$, and $t \geq 0$. Then

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$$J(f \star_t \rho)^{-1} - J(\rho)^{-1} - tJ(f)^{-1} \geq 0$$

Proof

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$$\mathrm{Tr}\left(J(f^{(\omega_c \vec{\theta})} \star_t \rho^{(\omega_q \vec{\theta})}); \vec{\theta}) \Big|_{\vec{\theta} = \vec{\theta}_0} \right) \leq \mathrm{Tr}\left(J(f^{(\omega_c \vec{\theta})}; \vec{\theta}) \Big|_{\vec{\theta} = \vec{\theta}_0} \right) + \mathrm{Tr}\left(J(\rho^{(\omega_q \vec{\theta})}; \vec{\theta}) \Big|_{\vec{\theta} = \vec{\theta}_0} \right)$$

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$$\omega^2 J(f \star_t \rho) \qquad \omega_c^2 J(f) \qquad \omega_q^2 J(\rho)$$

Taking $\omega_c = \frac{\sqrt{t} J(f)^{-1}}{J(\rho)^{-1} + tJ(f)^{-1}}$ and $\omega_q = \frac{J(\rho)^{-1}}{J(\rho)^{-1} + tJ(f)^{-1}}$, we obtain

$$J(f \star_t \rho)^{-1} - J(\rho)^{-1} - tJ(f)^{-1} \geq 0 \qquad \square$$

Concavity of entropy power of heat diffusion semigroup

Classical heat diffusion semigroup

$$\frac{\partial}{\partial t} f_t = \Delta f_t \qquad \text{where } \Delta = \sum_j \frac{\partial^2}{\partial x_j^2}$$

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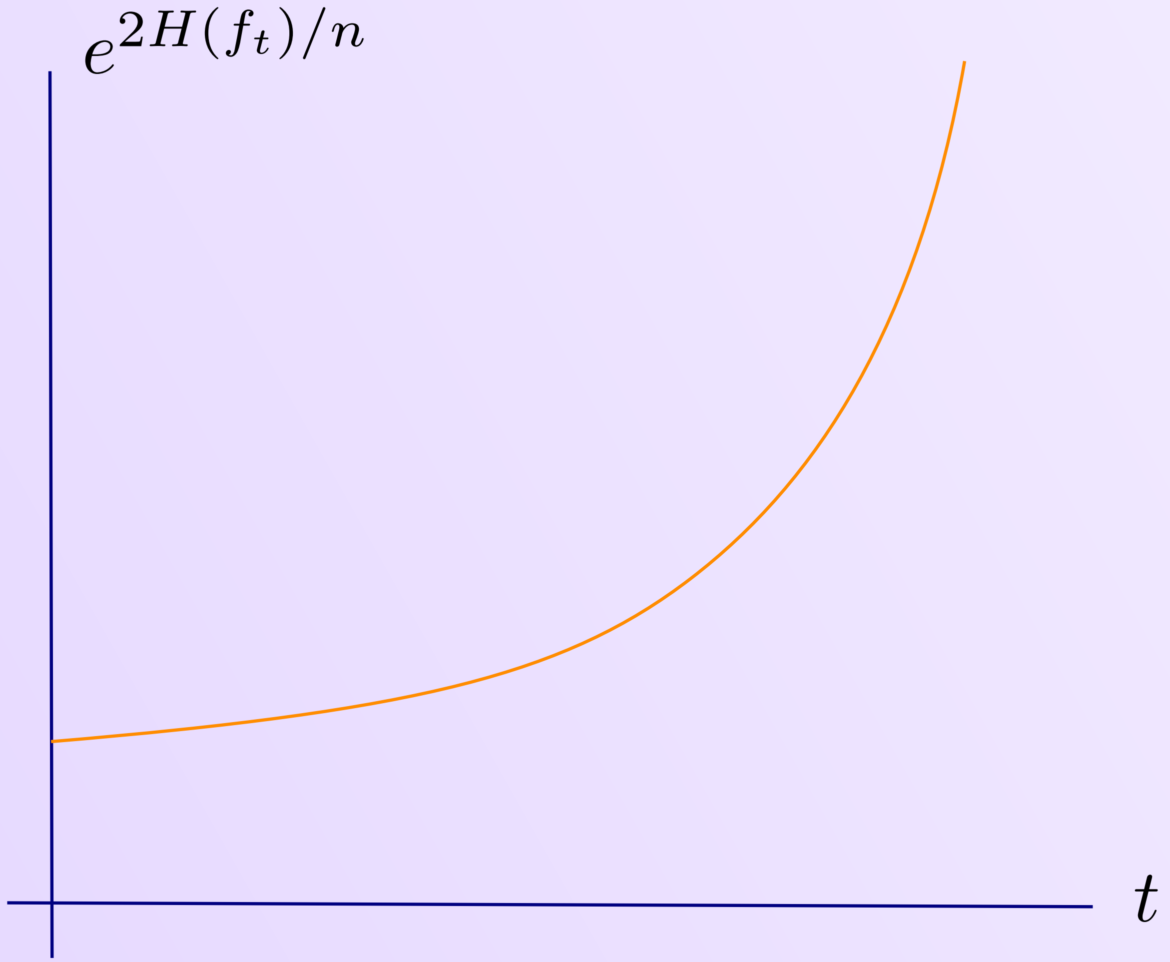
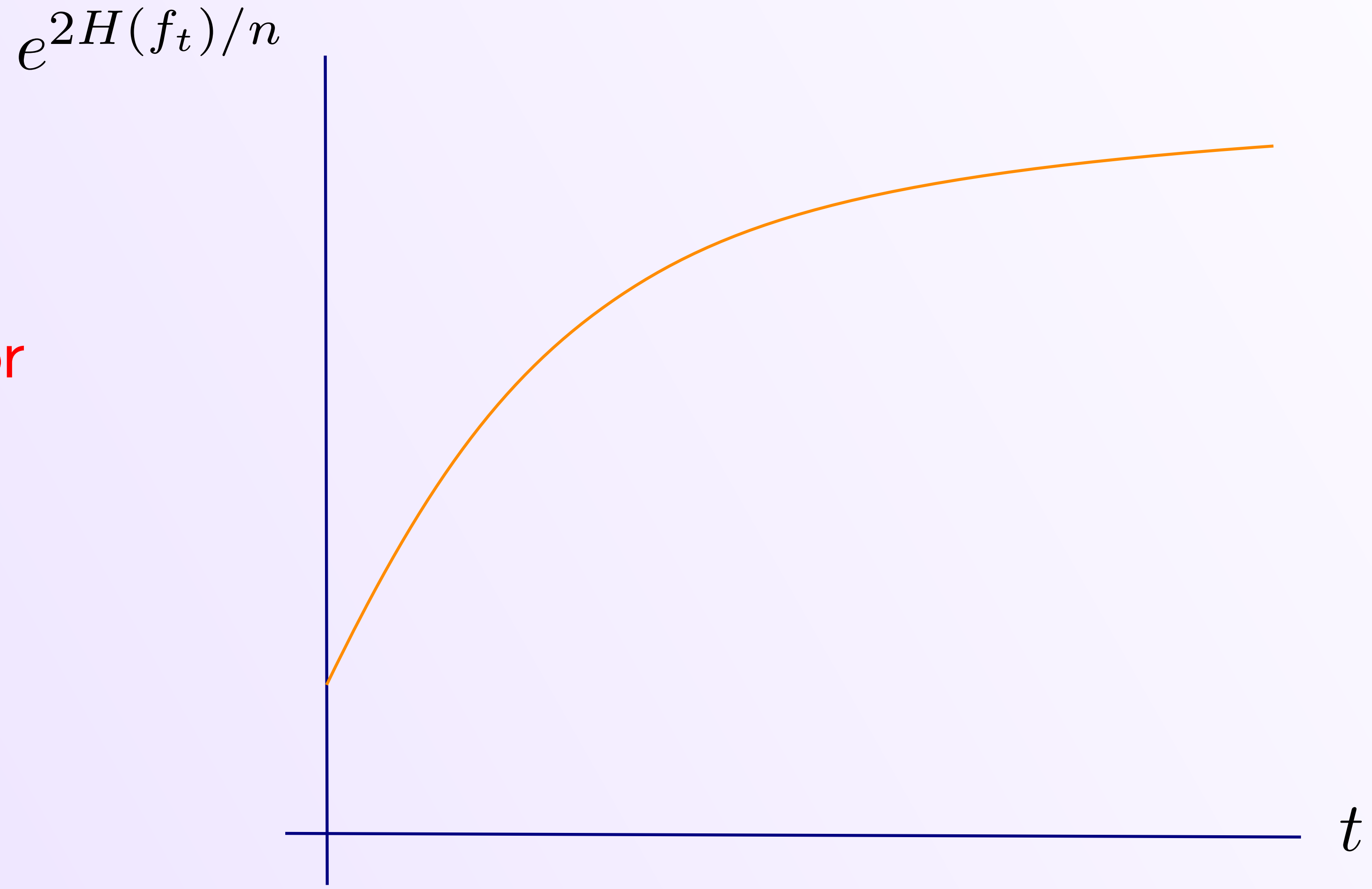


or



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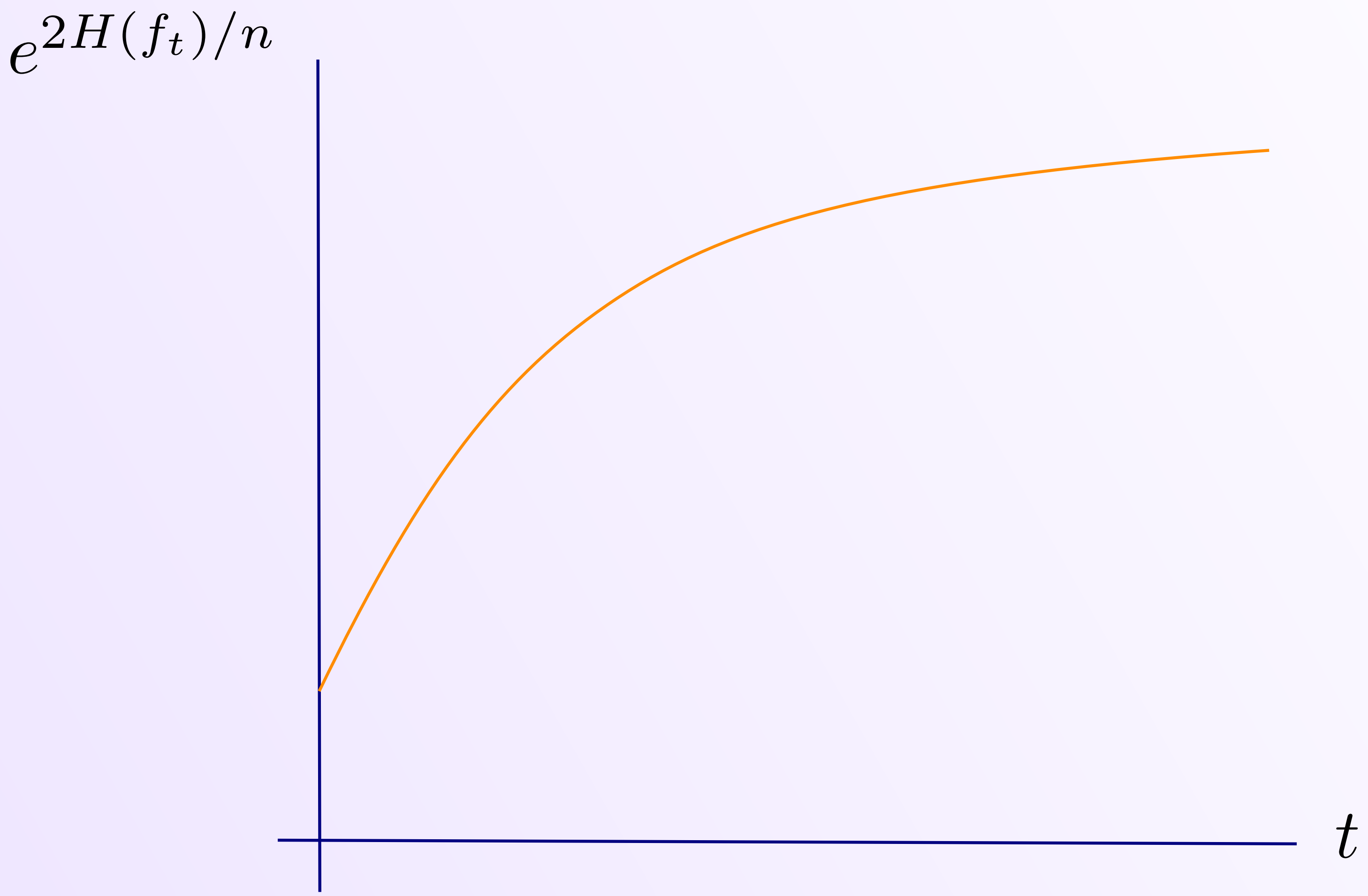
Theorem

$$\frac{d^2}{dt^2} e^{2H(f_t)/n} \leq 0$$

Dembo, Thomas, Cover '91

Villan '00



Concavity of entropy power of diffusion semigroup

Quantum diffusion semigroup

$\frac{d}{dt}\rho_t = \mathcal{L}(\rho_t)$ with $\mathcal{L}(\rho) = -\pi \sum_{j=1}^{2n} [Q_j, [Q_j, \rho]] + [P_j, [P_j, \rho]]$

Concavity of entropy power of diffusion semigroup

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Theorem

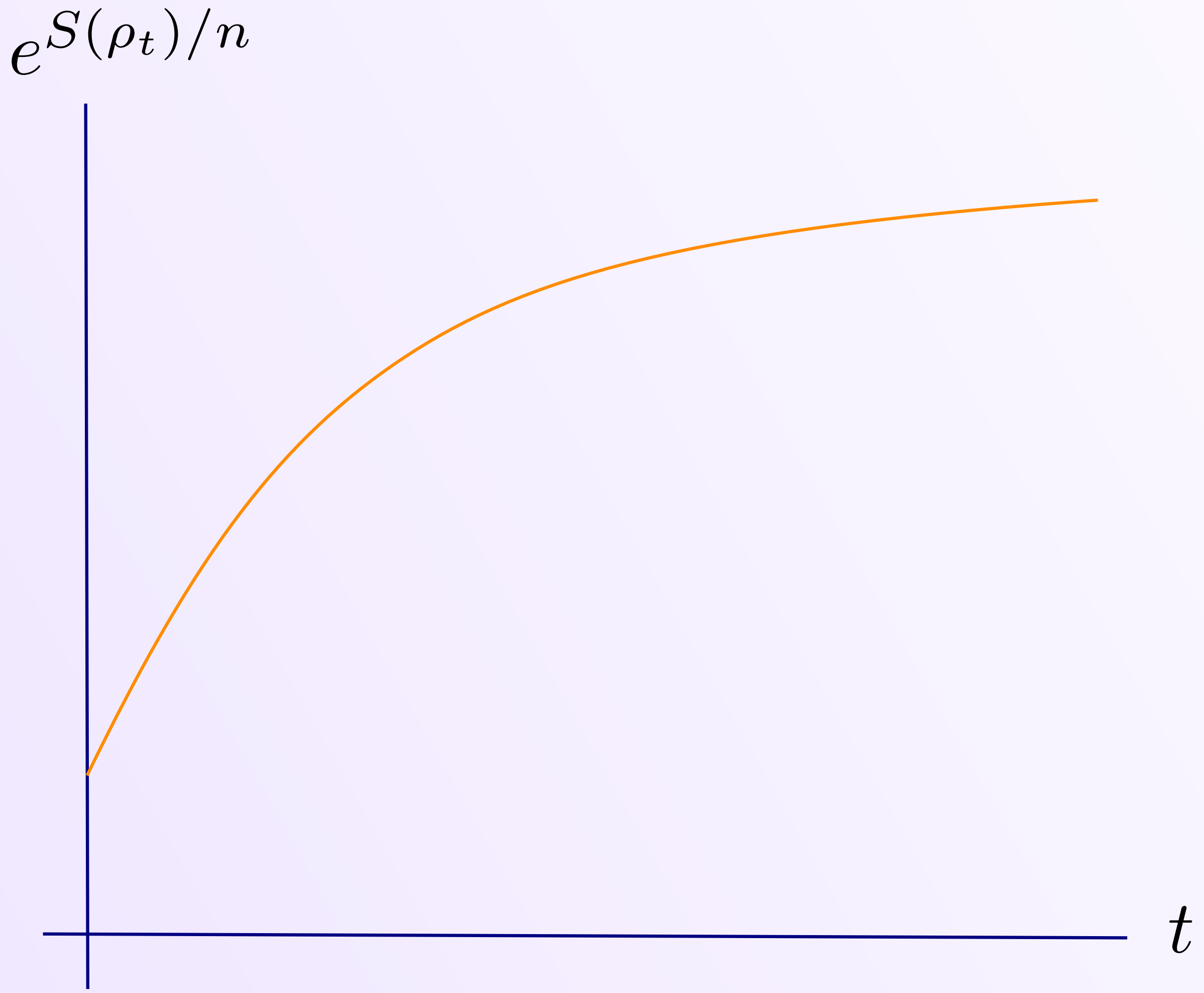
$$\left.\frac{d^2}{dt^2}\right|_{t=0} N(e^{t\mathcal{L}}(\rho)) \leq 0$$

Recall that for a Gaussian r.v. Z we have

$$e^{t\mathcal{L}}(\rho) = f_Z \star_t \rho$$

Therefore,

$$\left.\frac{d^2}{dt^2}\right|_{t=0} N(f_Z \star_t \rho) \leq 0$$



Quantum Fisher information isoperimetric inequality

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Proof: Take $f = f_Z$ in q. Stam inequality:

$$\frac{1}{t}\left\{ J(f_Z \star_t \rho)^{-1} - J(\rho)^{-1} \right\} \geq J(f_Z)^{-1} = (2n)^{-1}$$

Take a limit $t \to \infty$.

$\square$

Entropy power inequality

$$N(f \star_t \rho) \leq tN(f) + N(\rho)$$

$$e^{S(f \star_t \rho)/n} \leq te^{2H(f)/n} + e^{S(\rho)/n}$$

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In particular,

$$N(e^{t\mathcal{L}}(\rho)) \geq N(\rho) + t\, 2\pi e$$

Isoperimetric Inequality for Entropies

$$\frac{1}{n}J(\rho)N(\rho) \geq 4\pi e$$

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$$\frac{1}{n}J(\rho)N(\rho) \geq 4\pi e$$

Proof:

from de Bruijn identity:

$$\left.\frac{d}{dt}\right|_{t=0} N(\rho(t)) = \frac{1}{2n}J(\rho)N(\rho)$$

denoting $\rho(t) = e^{t\mathcal{L}}(\rho)$, $t \geq 0$, the entropy power inequality yields:

$$\frac{1}{t}[N(\rho(t)) - N(\rho)] \geq 2\pi e$$

take a limit $t \to \infty$.

$\square$

Isoperimetric Inequality for Entropies

$$\frac{1}{n} J(\rho) N(\rho) \geq 4\pi e$$

Optimality: let $\rho = \omega_{\mathbf{n}}$ be a Gaussian thermal state with a mean photon number $\mathbf{n}$

Then $S(\omega_{\mathbf{n}}) = g(\mathbf{n}) = (\mathbf{n}+1)\log(\mathbf{n}+1) - \mathbf{n}\log\mathbf{n}$

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Under $\mathcal{L}$ the state evolves as $e^{t\mathcal{L}}(\omega_{\mathbf{n}}) = \omega_{\mathbf{n}_t}$ where $\mathbf{n}_t = \mathbf{n} + 2\pi t$

In particular, by de Bruijn identity

$$J(\omega_{\mathbf{n}}) = 2 \left.\tfrac{d}{dt}\right|_{t=0} S(e^{t\mathcal{L}}(\omega_{\mathbf{n}})) = 4\pi \log\left(\tfrac{\mathbf{n}+1}{\mathbf{n}}\right)$$

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Also

$$N(\omega_{\mathbf{n}}) = \exp(S(\omega_{\mathbf{n}})/1) = \frac{(\mathbf{n}+1)^{\mathbf{n}+1}}{\mathbf{n}^{\mathbf{n}}}$$

Therefore

$$J(\omega_{\mathbf{n}})N(\omega_{\mathbf{n}}) = 4\pi \left(\frac{\mathbf{n}+1}{\mathbf{n}}\right)^{\mathbf{n}} \log\left(\frac{\mathbf{n}+1}{\mathbf{n}}\right)^{\mathbf{n}+1} \underset{\mathbf{n}\to\infty}{\longrightarrow} 4\pi e$$

Independent results by Rouzé, Datta, Pautrat

Quantum Nash Inequality For n-mode Gaussian states $\{\rho_j\}_1^N$, and $\lambda_j \in \mathbb{C}$, let $A = \sum_j \lambda_j \rho_j$

Then

$$\|A\|_2^{2+2/n} \leq -C\|A\|_1^{2/n}\mathrm{Tr}(A^*\mathcal{L}(A))$$

Non-commutative ultracontractivity There exists $\kappa_n > 0$, s.t. for any $t > 0$:

$$\|e^{t\mathcal{L}}\|_{1\to 2} \leq \kappa_n t^{-n/2}$$

here $\|\Lambda\|_{p\to q} = \sup_{\|A\|_p=1}\|\Lambda(A)\|_q$

$$\|e^{t\mathcal{L}}\|_{1\to\infty} \leq 2^n\kappa_n^2 t^{-n}$$

$\|A\|_p^p = \mathrm{Tr}(|A|^p)$

Therefore, $S(e^{t\mathcal{L}}(\rho)) \geq n\log(2\pi\kappa_n^{-\frac{2}{n}}t)$ $\quad$ $\mathrm{Tr}(\{e^{t\mathcal{L}}(\rho)\}^2) \leq \kappa_n^2 t^{-n}$

Quantum Blachman-Stam inequality For any $\alpha, \beta > 0$ and $t > 0$,

$$(\alpha+\beta)^2 J(e^{t\mathcal{L}}(\rho)) \leq \alpha^2 J(\rho) + \frac{2n\beta^2}{t}$$

Follows directly from Stam Inequality

Quantum Ornstein-Uhlenbeck semigroup

$n = 1$

Quantum Attenuator:

$$\mathcal{L}_-(\rho) = a\rho a^\dagger - \frac{1}{2}\{a^\dagger a, \rho\}$$

Quantum Amplifier:

$$\mathcal{L}_+(\rho) = a^\dagger \rho a - \frac{1}{2}\{aa^\dagger, \rho\}$$

Let $\rho = \omega_{\mathbf{n}}$ be a Gaussian thermal state with a mean photon number $\mathbf{n}$. Then $\rho_\pm(t) = \omega_{\mathbf{n}_\pm(t)}$

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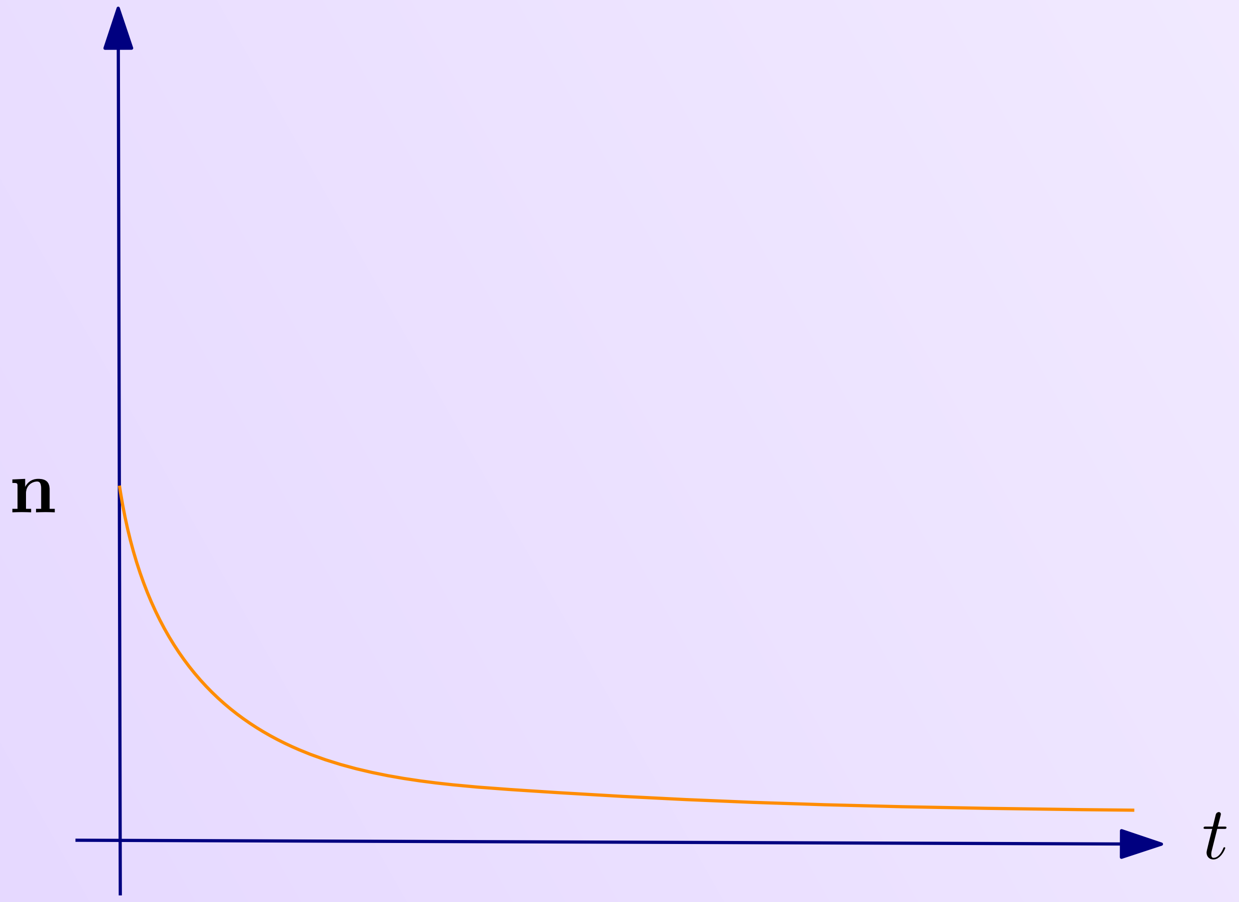
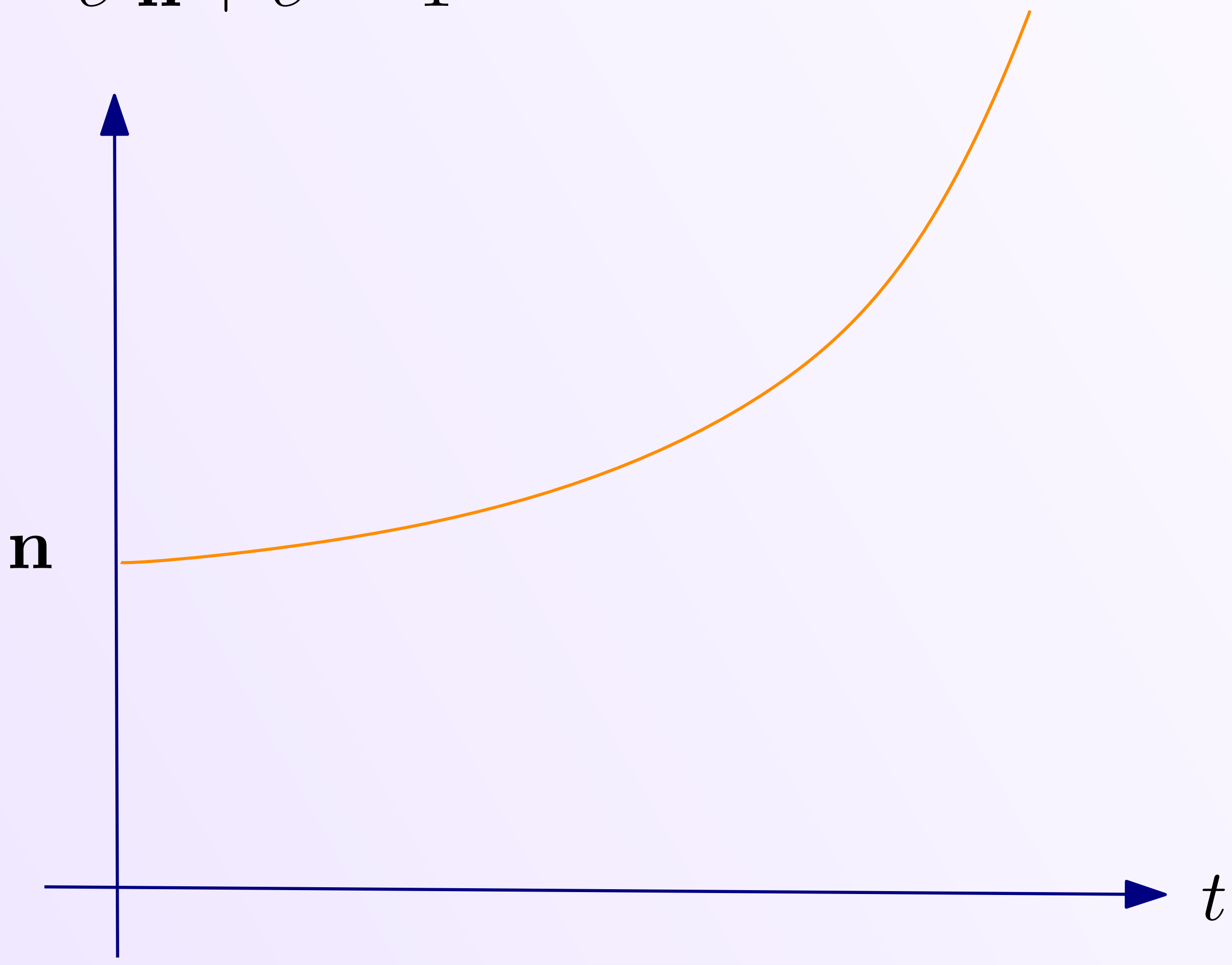
Under $\mathcal{L}_-$

$$\mathbf{n}_-(t) = e^{-t}\mathbf{n}$$



Under $\mathcal{L}_+$

$$\mathbf{n}_+(t) = e^t\mathbf{n} + e^t - 1$$



Quantum Ornstein-Uhlebeck semigroup

One-parameter group of CPTP maps $\{e^{\mathcal{L}_{\mu,\lambda}}\}_{t\geq 0}$, generated by

$$\mathcal{L}_{\mu,\lambda} = \mu^2\mathcal{L}_- + \lambda^2\mathcal{L}_+ \qquad \text{for } \mu > \lambda > 0$$

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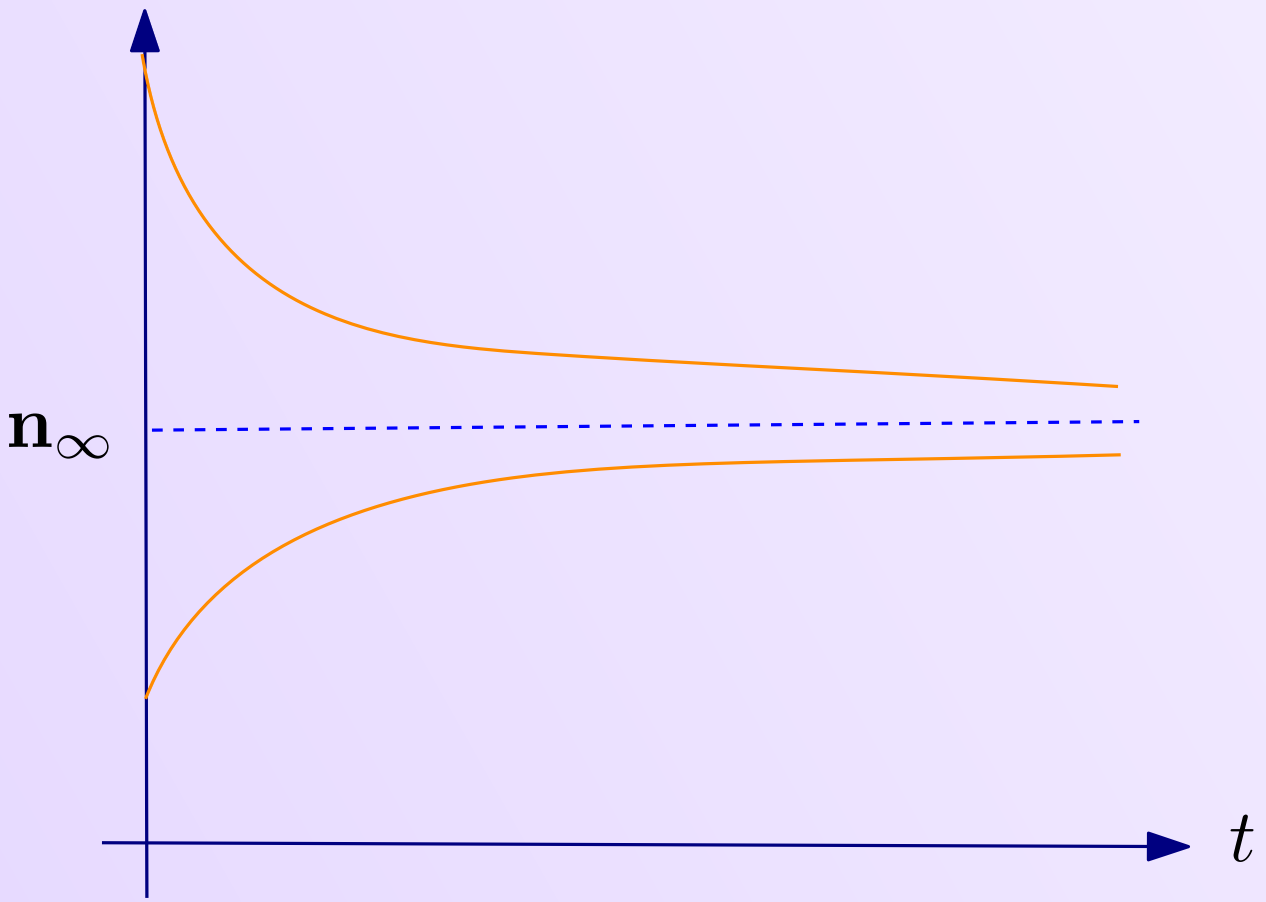
Then $\omega_{\mathbf{n}(t)}$ is s.t.

$$\mathbf{n}(t) = e^{-(\mu^2-\lambda^2)t}\mathbf{n} + (1-e^{-(\mu^2-\lambda^2)t})\mathbf{n}_\infty \qquad \text{with } \mathbf{n}_\infty = \mathrm{tr}(\sigma_{\mu,\lambda}\hat{n}) = \frac{\lambda^2}{\mu^2-\lambda^2}$$



Fast convergence of qOU semigroup

Theorem

For any one-mode Gaussian state ρ we have

$$D(e^{t\mathcal{L}_{\mu,\lambda}}(\rho)\|\sigma_{\mu,\lambda}) \leq e^{-(\mu^2-\lambda^2)t} D(\rho\|\sigma_{\mu,\lambda}) \qquad \text{for all } t \geq 0$$

In other words,

$$\frac{d}{dt}\Big|_{t=0} D(e^{t\mathcal{L}_{\mu,\lambda}}(\rho)\|\sigma_{\mu,\lambda}) \leq -(\mu^2-\lambda^2) D(\rho\|\sigma_{\mu,\lambda})$$

Moreover, for any $\zeta > \mu^2 - \lambda^2$ there exists a Gaussian state ρ s.t.

$$\frac{d}{dt}\Big|_{t=0} D(e^{t\mathcal{L}_{\mu,\lambda}}(\rho)\|\sigma_{\mu,\lambda}) > -\zeta D(\rho\|\sigma_{\mu,\lambda})$$

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Conjecture (June '16)

For any state ρ we have

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Proved by Calren, Mass in Sep '16

Quantum Log-Sobolev inequality

From Isoperimetric inequality for entropies for $A > 0$

$$-S(\rho) \leq AJ(\rho) - (2 + \log(4\pi A))$$

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$$-S(\rho) \leq AJ(\rho) - (2 + \log(4\pi A))$$

$$-S(\rho) \leq \log\left\{\frac{1}{4\pi e}J(\rho)\right\}$$

$$\log\left(\frac{1}{4\pi e}J(\rho)\right) = \log\left(\frac{AJ(\rho)}{4\pi e A}\right)$$

$$= \log\left(\frac{1}{4\pi e A}\right) + \log\left(AJ(\rho)\right)$$

using $\log x \leq x - 1$

$$\leq AJ(\rho) - 2 - \log(4\pi A)$$

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From Isoperimetric inequality for entropies for $A > 0$

$$-S(\rho) \leq AJ(\rho) - (2 + \log(4\pi A))$$

Note that

$$D(\rho \| \sigma_{\mu,\lambda}) = -S(\rho) - (\log \nu)\mathbf{n} - \log(1-\nu) \qquad \nu = \lambda^2/\mu^2 < 1$$

Therefore

$$D(\rho \| \sigma_{\mu,\lambda}) \leq AJ(\rho) - 2 - \log(4\pi A) + \mathbf{n} \log \frac{1}{\nu} - \log(1-\nu)$$

Classical Log-Sobolev inequality

[Carlen '91]

Gross' logarithmic Sobolev inequality is

$$\int |f|^2 \log |f|^2 e^{-\pi|x|^2} dx \leq \frac{1}{\pi} \int |\bigtriangledown f|^2 e^{-\pi|x|^2} dx \qquad \text{for } \int |f|^2 e^{-\pi|x|^2} dx = 1$$

Let $g(x) = f(x)e^{-\pi|x|^2/2}$. Then

$$\int |g|^2 \log |g|^2 dx \leq \frac{1}{\pi} \int |\bigtriangledown g|^2 dx - n \qquad \text{with } \int |g|^2 dx = 1$$

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$$H(X|Y)$$

$$\int h_X \log h_X dx - \int h_X \log h_Y dx$$

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Quantum case:

$$D(\rho \| \sigma_{\mu,\lambda}) \leq A J(\rho) - 2 - \log(1-\nu) - \log(4\pi A) + \mathbf{n} \log \frac{1}{\nu} \qquad \nu = \lambda^2/\mu^2 < 1$$

$$A > 0$$

Fast convergence of Ornstein-Uhlenbeck semigroup

$$-\frac{d}{dt}\Big|_{t=0} D(e^{t\mathcal{L}_{\mu,\lambda}}(\rho)\|\sigma_{\mu,\lambda}) - (\mu^2 - \lambda^2) D(\rho\|\sigma_{\mu,\lambda}) \geq 0$$

Fast convergence of Ornstein-Uhlenbeck semigroup

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$$J_{\pm}(\rho) := 2\frac{d}{dt}S(e^{t\mathcal{L}_{\pm}}(\rho))$$

$$\frac{\mu^2}{2}J_{-}(\rho) + \frac{\lambda^2}{2}J_{+}(\rho) - (\log\nu)(\mu^2 - \lambda^2)\mathbf{n} + \lambda^2\log\nu$$

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[de Palma et al. '16]:
$J_-(\rho) \geq f(S(\rho))$

[Buscemi et al. 16]:
$J_+(\rho) \geq 2$

For large $S(\rho)$, e.g. $\mu^2 = 2$, $\lambda^2 = 1$: $S(\rho) \gtrsim 0.5$

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isoperimetric ineq.

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Note that $J(\rho) = 2\pi(J_-(\rho) + J_+(\rho))$

Take $A = \frac{\lambda^2}{4\pi(\mu^2-\lambda^2)}$

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Gaussian optimality:

$$\geq J_-(\omega_{\mathbf{n}}) = -2\mathbf{n}\log(1 + 1/\mathbf{n})$$

For small $\mathbf{n} = \mathrm{Tr}(\rho\hat{n})$, e.g. $\mu^2 = 2$, $\lambda^2 = 1$: $\mathbf{n} \lesssim 0.67$

Thank you!

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Gaussian optimality for energy-constrained entropy rates

Quantum Attenuator: $\mathcal{L}_-(\rho) = a\rho a^\dagger - \frac{1}{2}\{a^\dagger a, \rho\}$

Theorem

$$\inf_{\rho:\mathrm{Tr}(\hat{n}\rho)\leq \mathbf{n}} \frac{d}{dt}\Big|_{t=0} J_-(\rho)) = J_-(\omega_{\mathbf{n}}) = -\mathbf{n}\log\left(1+\frac{1}{\mathbf{n}}\right)$$

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Step 1 (Correspondence to classical problem)

Let $p = (p_0, p_1, \ldots,)$ be a prob. distribution

For $\rho = \sum_n p_n |n\rangle\langle n|$ we have

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with $\dot{p}_n(t) = -np_n(t) + (n+1)p_{n+1}(t)$

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Then $p(t) = e^{t\mathcal{C}_-}(p)$ - pure-death process

Theorem

$$\inf_{\rho:\mathrm{Tr}(\hat{n}\rho)\leq \mathbf{n}} \frac{d}{dt}\Big|_{t=0} S(e^{t\mathcal{L}_-}(\rho)) = \inf_{p:\mathbb{E}_p[N]\leq \mathbf{n}} \frac{d}{dt}\Big|_{t=0} H(e^{t\mathcal{C}_-}(p))$$

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Step 3 (Gaussian optimality)

Let $\omega_\mathbf{n}$ be a Gaussian thermal state with a mean-photon number $\mathbf{n}$

Then

$$\frac{d}{dt}\Big|_{t=0} S(e^{t\mathcal{L}_-}(\omega_\mathbf{n})) = -\mathbf{n}\log\left(1+\frac{1}{\mathbf{n}}\right)$$